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NAVAL WARFARE ANALYSIS GROUP

A MODEL OF CARRIER-SUBMARINE  
~~test~~ INTERACTIONS

By J.V. Hall, Cdr., USN  
J.K. Tyson, CNA  
J.S. Finucane, CNA

CNA Research Contribution No. 137

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1. Enclosure (1) is forwarded as a matter of possible interest.
2. This Research Contribution describes a parametric model of engagements between submarines and a single aircraft carrier with ASW support. The measures of effectiveness are the probability that the carrier can remain on station for a specified length of time and the expected number of submarines killed. The model was developed as an aid in the study of how various tactics and forces can improve the efficiency of carrier operations.
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*James K. Tyson*  
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CNA RESEARCH CONTRIBUTION NO. 137

# NAVAL WARFARE ANALYSIS GROUP

CENTER FOR NAVAL ANALYSES

## A MODEL OF CARRIER-SUBMARINE INTERACTIONS

By J. V. Hall, Cdr., USN  
J. K. Tyson, CNA  
J. S. Finucane, CNA

February 1970

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NAVVAG ASW MODELS AND FORCES STUDY

K.H. Bohlin, Project Director  
J.V. Hall, Cdr., USN, Project Officer

## ABSTRACT

This paper develops analytic expressions for the probability that a single aircraft carrier, opposed by submarines, can remain on station for a specified length of time. The threat from both torpedo and cruise-missile submarines is considered, but the threat from aircraft is not. Expressions for expected submarine losses are derived. Expressions are also developed to show how the probability of remaining on station improves as the carrier's resistance to damage increases. A wide variety of ASW force compositions and tactics can be represented in the parameters of the model. Appendix C presents an alternative way to formulate the problem and derive the quantities mentioned above. The two methods produce results that are in excellent agreement. The intention of the authors is to provide a tool that can be used to improve carrier effectiveness through the study of tactics and force interactions.

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#### **ACKNOWLEDGEMENT**

This research contribution is partially based on ideas developed in earlier work by Dr. Robert D. Arnold, Dr. Erwin Baumgarten, and Dr. James K. Tyson, of the Center for Naval Analyses.

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## A MODEL OF CARRIER-SUBMARINE INTERACTIONS

### INTRODUCTION

When submarines oppose a carrier that is operating in the same general area for a period of time, the events that take place are very complex and cannot be exactly described in mathematical terms. This paper defines and models a simple process that is similar in some respects to the above events. Suitably interpreted, the model can be used to study the interactions of a compact or dispersed carrier formation and torpedo or cruise-missile submarines.

The following rules govern the modeled process. (1) There is a constant rate of encounter<sup>1</sup> (probability of encounter per unit time) between the carrier and each surviving submarine. (2) At each encounter between the carrier and a submarine there is a fixed probability that the carrier will be successfully attacked<sup>2</sup> and a fixed probability the submarine will be killed. (3) As long as the carrier remains on station, the rate of attrition to each surviving submarine (probability of kill per surviving submarine per unit time) is constant. When the carrier retires all action stops and all rates become zero. (4) The submarines are identical and their operations are mutually independent.

The following notation is used in the model:

- a = Rate at which surviving submarines are attrited when there are no successful attacks on the carrier ("submarines killed per surviving submarine per unit of time")
- b = Rate of encounter between each surviving submarine and the carrier ("encounters per surviving submarine per unit time")
- h = Pr (carrier is successfully attacked, given an encounter)
- $\alpha$  = Rate at which surviving submarines are attrited in the absence of encounters with the carrier.

---

<sup>1</sup> Definitions of "encounter" for various situations are given in the "Discussion" section.

<sup>2</sup> A "successful attack" is an attack resulting in one or more hits on the carrier.

- $s$  = Unconditional rate at which surviving submarines are attrited ("submarines killed per surviving submarine per unit time")  
 $N$  = Number of successful attacks the carrier can absorb before being forced to retire  
 $x_0$  = Initial number of attacker's submarines  
 $r$  = Rate at which additional submarines arrive in the area  
 $K(t)$  = Expected number of submarines killed up to time  $t$   
 $R(t)$  = Ratio of expected number of submarines killed to the probability that the carrier has been forced to retire up to time  $t$   
 $P_n(t)$  = Pr (carrier has been successfully attacked exactly  $n$  times by time  $t$ ,  $n=0, 1, 2, \dots, N$ )  
 $\pi_n(t)$  = Pr (carrier has been successfully attacked  $n$  or fewer times by time  $t$ ,  $n=0, 1, 2, \dots, N$ ).

The first 8 quantities listed are inputs to the model; the last 4 are outputs.

#### SUMMARY OF EQUATIONS DERIVED

The equations summarized in this section are derived in appendix A.

The probability that the carrier has not been successfully attacked by time  $t$  is

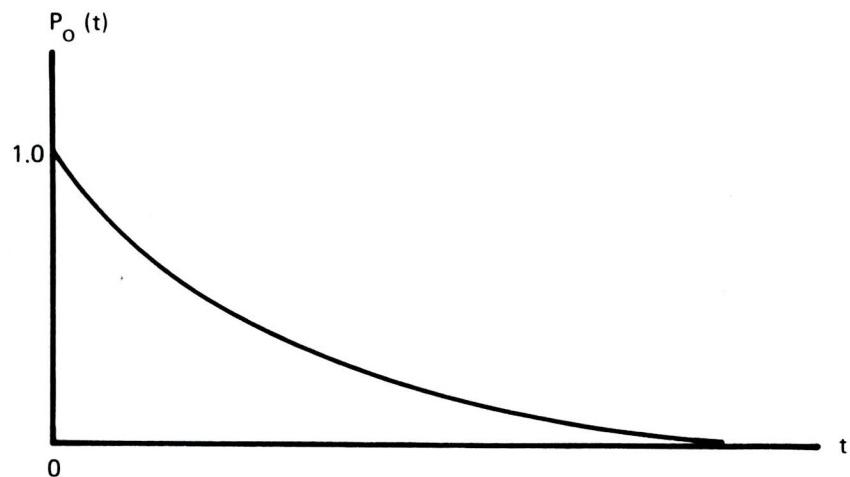
$$P_o(t) = \exp \left[ -\frac{bh}{a} \left( rt + (x_0 - r/a)(1 - e^{-at}) \right) \right]. \quad (1)$$

This function has the general shape shown in figure 1. A special case occurs when  $r=0$ , meaning that no additional submarines arrive in the area after the action begins. Then

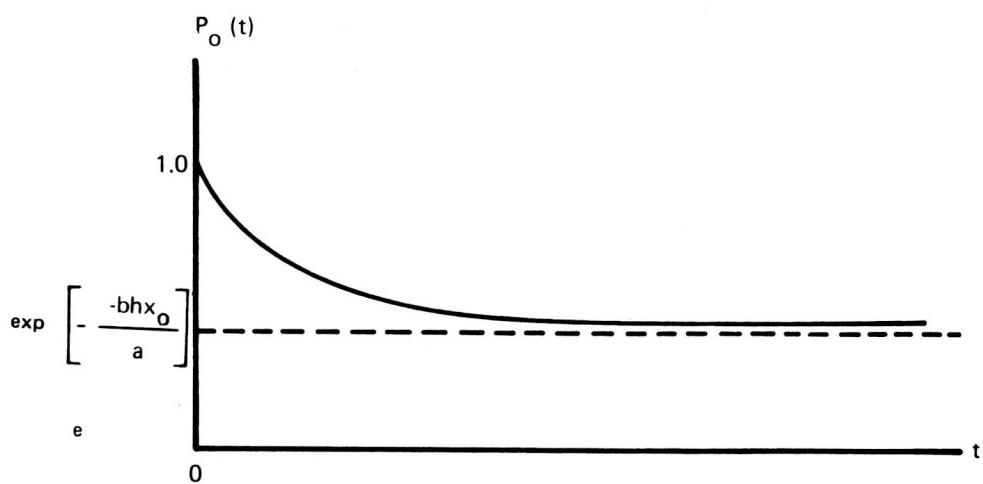
$$P_o(t) = \exp \left[ -\frac{bx_0}{a} (1 - e^{-at}) \right] \quad (2)$$

and the function (shown in figure 2) has a limiting value given by

$$\lim_{t \rightarrow \infty} P_o(t) = \exp \left[ -\frac{bx_0}{a} \right]. \quad (3)$$



**FIGURE 1**



**FIGURE 2**

This is the probability that, in the special case where  $r=0$ , the carrier will not be successfully attacked before all of the submarines,  $x_0$  in number, are killed.

For a carrier that can absorb  $N$  successful attacks before being forced to retire, the probability of undergoing exactly  $n$  successful attacks by time  $t$  is, for  $n \leq N$ ,

$$P_n(t) = P_0(t) \frac{[-\ln P_0(t)]^n}{n!} . \quad (4)$$

The probability that such a carrier has undergone  $n$  or fewer successful attacks is, for  $n \leq N$ ,

$$\pi_N(t) = \sum_{k=0}^n P_k(t) . \quad (5)$$

If the carrier can absorb  $N$  successful attacks before being forced to retire, its probability of being on station at any time  $t$  is  $\pi_N(t)$ . This probability is shown in figure 3, plotted against  $1 - P_0(t)$ ; that is, figure 3 shows the probability that the carrier is still on station versus the probability that the carrier has been successfully attacked at least once. The feature of figure 3 is that it shows the relative advantage enjoyed by a carrier that can absorb some successful attacks.

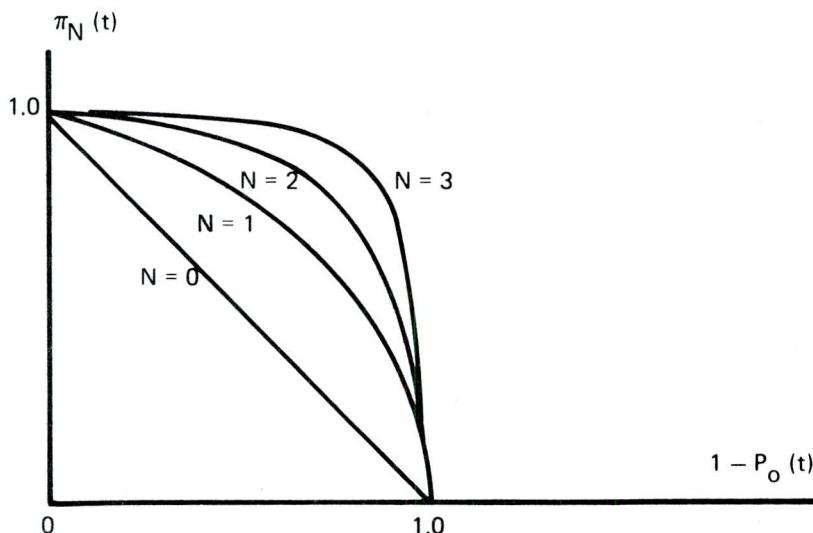


FIGURE 3

The expected number of submarines that are killed up to time  $t$  is given by

$$K(t) = \frac{S}{bh} \sum_{k=0}^N [1 - \pi_k(t)]. \quad (6)$$

The "exchange ratio" is given by

$$R(t) = \frac{K(t)}{1 - \pi_N(t)}. \quad (7)$$

If the carrier cannot absorb any successful attacks and must retire the first time it is hit, then  $N=0$ , and the exchange ratio is independent of the variable  $t$ :

$$R(t) = \frac{S}{bh}. \quad (8)$$

For  $N \geq 1$ , it is possible to calculate the expected number of submarines the enemy would have to expend to force the carrier to retire, if he were willing to commit submarines for as long as necessary:

$$\lim_{t \rightarrow \infty} R(t) = (N+1) \frac{S}{bh}. \quad (9)$$

It should be noted that equations 8 and 9 contain no mention of the enemy's initial commitment ( $x_0$ ) or of the rate at which additional submarines are committed ( $r$ ).

## DISCUSSION

This model involves aggregated parameters, each of which has an obvious physical interpretation. The appropriate values to use in any particular case must be determined by separate analysis.

The encounter rate ( $b$ ) depends on whether the submarines are armed with missiles and torpedoes or torpedoes only. For a missile submarine "encounter" occurs when the carrier has been localized closely enough that the commanding officer feels he has enough information to warrant firing. For a torpedo submarine, "encounter" does not occur until the submarine has approached the screen closely enough to risk detection.<sup>3</sup>

---

<sup>3</sup> If there is no screen, "encounter" occurs when the submarine reaches firing position.

The encounter rate also depends on the amount of information available to the submarines. If they must search at random for the carrier, an encounter rate can be derived from the average detection range, the speeds of carrier and submarines, size of the operating area, and the weapon firing range.<sup>4</sup> The detection range depends on how much electromagnetic and acoustic noise is radiated by the carrier and its close escorts. If the carrier is very quiet and can be identified only by visual sighting, it may be appropriate to take account of data suggesting that submarines have a near-zero capability in this regard during darkness.<sup>5</sup> (It may also be better for missile submarines to fire torpedoes in this case.) If the submarines have some external intelligence so that their search is not random, the encounter rate must be selected to reflect the details of the particular situation. For practical purposes, if the carrier or its close escorts are very noisy so that the submarines can achieve very long average detection ranges with their own sensors, it is the same as if they had external intelligence. The encounter rate in such cases may be limited by a submarine's ability to maneuver and its boldness, rather than by its ability to localize the carrier.

Attrition rate  $\alpha$  results from the action of all the ASW forces used in any particular case. It is the rate of kill that occurs in the absence of successful attacks on the carrier;<sup>6</sup> hence, it incorporates the kills that occur when submarines encounter the carrier and try unsuccessfully to attack it, but excludes kills resulting from successful attacks.<sup>7</sup> It can be seen that the attrition rate depends on the encounter rate, since each encounter offers some risk to the submarine as well as the carrier. The remaining component of  $\alpha$  is the rate ( $\alpha$ ) at which surviving submarines are attrited in the absence of encounters with the carrier. The attrition rate, then, also includes kills by ASW aircraft and other systems that may be present in a particular case.

The attrition rate is one of three parameters in this model that incorporate estimates of ASW force effectiveness. The others are the encounter rate (in their encounter attempts submarines may be slowed down and otherwise inhibited by effective ASW forces), and the probability of successful attack given an encounter ( $h$ ).

<sup>4</sup> Reference (a), page 7 ff.

<sup>5</sup> References (b) and (c).

<sup>6</sup> This definition of  $\alpha$  is a consequence of the way equation 1 is derived (see appendix A).

<sup>7</sup> Kills resulting from successful attacks are accounted for in equations 6 through 9.

Values for  $h$  can be obtained from analysis of a submarine's attack after it has encountered the carrier. For a missile submarine,  $h$  involves AAW as well as ASW considerations. For a torpedo submarine,  $h$  involves screen penetration probabilities and carrier evasion probabilities. The probability that the submarine is killed given that it makes a successful attack also involves different factors for the two different types of submarines.

A convenient way to summarize the possible outcomes of an encounter, no matter which kind it is, is as follows:

		Carrier hit	
		Yes	No
Submarine killed	Yes	$P_1$	$P_2$
	No	$P_3$	$P_4$

In every encounter, the carrier is either hit or not hit and the submarine is either killed or not killed.<sup>8</sup> Thus:

$$\begin{aligned} P_1 &= \text{Pr}(\text{carrier hit and submarine killed}) \\ P_2 &= \text{Pr}(\text{carrier not hit and submarine killed}) \\ P_3 &= \text{Pr}(\text{carrier hit and submarine not killed}) \\ P_4 &= \text{Pr}(\text{carrier not hit and submarine not killed}). \\ P_1 + P_2 + P_3 + P_4 &= 1. \end{aligned}$$

Parameters  $h$ ,  $a$ , and  $S$ , required by the model, are related to these probabilities as follows:

$$\begin{aligned} h &= P_1 + P_3 \\ a &= bP_2 + \alpha \\ S &= b(P_1 + P_2) + \alpha. \end{aligned}$$

The number of successful attacks ( $N$ ) a carrier can absorb before being forced to retire depends on the number of hits per successful attack and the number of hits the carrier can take.<sup>9</sup> These numbers are, in general, not the same for missiles as for torpedoes.<sup>9</sup> In any event,  $N$  will probably be based as much on judgment as on analysis.  $N$  can also be varied to study the question "what would happen if the carrier were constructed so as to be able to absorb 1, 2, etc., successful attacks."

<sup>8</sup> These kills include all prosecutions triggered by the encounter.

<sup>9</sup> References (d), (e), (f), and (g) contain data on the vulnerability of carriers to missile or torpedo hits.

In summary, the parameters of this model are determined by —

- The number, type, and effectiveness of the enemy's submarines,
- The intelligence available to the enemy's submarines,
- The number, type and effectiveness of the ASW forces,
- The details of the situation and the tactics employed, and
- The vulnerability of the carrier.

As these change, they cause the parameters of the model to change. Therefore, the model can be used to study the ways in which the type of enemy submarine etc. affect carrier on-station probability and enemy submarine losses.

Appendix B gives some numerical examples to illustrate the application of this model.

## REFERENCES

- (a) OEG Report 56, "Search and Screening," Unclassified
- (b) OEG Study 642, "Submarine Opposition to Carriers in Large Dispersed Dispositions" (U), Secret
- (c) ComSeventhFlt (GH) ser. N31-00104 "SeventhFlt ASW Experience in SeventhFlt STRIKEX/ASWEX During 1961" (U), Secret 7 Jun 1962
- (d) USNRDL Technical Report 68-103, Secret/Noforn, Oct 1968
- (e) USNRDL Technical Report 68-132, Secret/Noforn, Nov 1968
- (f) WAS(NOW), vol. VA, part VII, Secret
- (g) "Probability of Damage to Ships from Bomb, Missile, and Torpedo Attack" (U), NSRDC, Confidential 3 Nov 1967
- (h) OEG IRM-22, "A Lanchester-Type Model for Combat Between Submarines, Carrier Task Groups, and Hunter-Killer Groups," J. Bram, Unclassified 28 Aug 1962

## APPENDIX A

## APPENDIX A

### MATHEMATICAL DERIVATIONS

This model treats the carrier-submarine problem by using differential equations. Five constant quantities described as "rates" are used in formulating the equations:  $a$ ,  $b$ ,  $\alpha$ ,  $S$ , and  $r$ . In some instances, it is useful to think these rates as "probabilities per unit time" that certain events will occur. In particular, when deriving numerical values of the parameters from data or theoretical analyses, it is often convenient to think in terms of probabilities and expected values, rather than rates. The conversion of probabilities to rates and vice versa is a satisfactory procedure, provided one selects a unit of time small enough that there is little chance of an event's occurring more than once during a unit of time, and yet large enough that there is some probability of its occurring once.

Let  $x(t)$  be the expected number of attacker's submarines alive at  $t$ , given the carrier has not been successfully attacked up to time  $t$ . (The other symbols used in this appendix are defined in the main body.)

If the carrier has not been successfully attacked up to time  $t$ , the probability that it will not be successfully attacked during an additional increment of time  $dt$  is given by

$$P_o(dt|t) = 1 - b x(t) h dt, \quad (A-1)$$

so that  $P_o(t+dt) = P_o(t) P_o(dt|t)$

$$= P_o(t) [1 - b x(t) h dt]$$

and  $dP_o(t) = P_o(t+dt) - P_o(t) = -bhx(t) P_o(t) dt . \quad (A-2)$

This leads to

$$\int_1^{P_o(t)} \frac{dP_o}{P_o} = -bh \int_0^t x(t) dt$$

and  $P_o(t) = \exp \left\{ -bh \int_0^t x(t) dt \right\}, \quad (A-3)$

where  $\int_0^t x(t) dt$  is simply the total number of search days achieved by all submarines up to time  $t$ , given the carrier has not been successfully attacked.

The expression for the expected number of submarines present,  $x(t)$ , may be written as

$$\frac{dx}{dt} = -ax + r, \quad (A-4)$$

$$\text{so that } x(t) = r/a + e^{-at} (x_0 - r/a). \quad (A-5)$$

When this expression for  $x(t)$  is used with equation A-3, the result is

$$P_o(t) = \exp \left[ \frac{-bh}{a} [rt + (x_0 - r/a)(1 - e^{-at})] \right]. \quad (A-6)$$

An important special case occurs when  $r=0$ , meaning that the enemy commits no additional submarines after the action begins. In this case

$$P_o(t) = \exp \left[ -\frac{bhx_0}{a} (1 - e^{-at}) \right]. \quad (A-7)$$

When  $r=0$  it is possible that all  $x_0$  of the submarines may be killed before the carrier is successfully attacked. The probability of this event is

$$\lim_{t \rightarrow \infty} P_o(t) = \exp \left[ -\frac{bhx_0}{a} \right]. \quad (A-8)$$

An expression for  $P_n(t)$ , the probability that the carrier has been successfully attacked exactly  $n$  times by time  $t$ , is derived as follows. Denote by  $P_o(t-T)$  the probability that there is no successful attack in the interval  $(T, t)$ , given that the carrier is on station at  $T$ . Then  $P_o(t-T)$  is given by equation A-6 except that the  $x_0$  of equation A-6 is replaced by  $x(T)$ , and  $t$ , by  $t-T$ ; that is, relative to the interval  $(T, t)$ , the initial number of submarines is  $x(T)$ .

Intuitively, it would be expected that

$$P_o(t-T) P_o(T) = P_o(t), \quad (A-9)$$

and by use of equation A-6 it is easy to show that this is correct. Now for  $n \leq N$ ,

$$P_n(t) = \int_{t_n=0}^t \cdots \int_{t_1=0}^{t_2} P_o(t-t_n) \left[ bh x(t_n) dt_n P(t_n - t_{n-1}) \right] \cdots \left[ bh x(t_1) dt_1 P_o(t_1) \right]. \quad (A-10)$$

Using equation A-9 reduces this to

$$P_n(t) = P_o(t) \int_{t_n=0}^t \dots \int_{t_1=0}^{t_2} b h x(t_n) dt_n \dots b h x(t_1) dt_1 . \quad (A-11)$$

From equation A-2

$$b h x(t_i) = -\frac{1}{P_o(t_i)} \frac{dP_o(t_i)}{dt_i} = -\frac{d(\ln P_o(t_i))}{dt_i} \quad (A-12)$$

Using equation A-12 in equation A-11 and integrating gives

$$P_n(t) = P_o(t) \frac{[-\ln P_o(t)]^n}{n!} . \quad (A-13)$$

Since this is the probability of exactly  $n$  successful attacks, the probability of  $N$  or fewer successful attacks by time  $t$  is given by

$$\pi_N(t) = \sum_{n=0}^N P_n(t) . \quad (A-14)$$

Next an expression is derived for  $K(t)$ , the expected number of submarines killed up to time  $t$ . From the definition of the process that is modeled here,

$$\frac{dK(t)}{dt} = \pi_N(t) S x(t) . \quad (A-15)$$

By use of equations A-12, A-13, and A-14 this can be written as

$$\frac{dK}{dt} = -\frac{S}{bh} \frac{dP_o}{dt} \sum_{n=0}^N \frac{[-\ln P_o]^n}{n!} . \quad (A-16)$$

$$\text{Therefore, } K(t) = -\frac{S}{bh} \sum_{n=0}^N \int_1^{P_o(t)} \frac{[-\ln P_o]^n}{n!} dP_o . \quad (A-17)$$

Carrying out the integration of an individual term gives

$$\int_1^{P_o(t)} \frac{[-\ln P_o]^n}{n!} dP_o = \pi_n(t) - 1 , \quad (A-18)$$

$$\text{so that } K(t) = \frac{S}{bh} \sum_{n=0}^N [1 - \pi_n(t)] . \quad (A-19)$$

The "exchange ratio"  $R(t)$  is defined by

$$R(t) = \frac{K(t)}{1 - \pi_N(t)} . \quad (A-20)$$

For  $N=0$ ,

$$R(t) = \frac{S}{bh} \quad (A-21)$$

for all values of  $t$ . For  $N > 0$ ,

$$\lim_{t \rightarrow \infty} R(t) = (N+1) \frac{S}{bh} , \quad (A-22)$$

provided that  $r > 0$ . The interpretation of equation A-22 was given in the main body.

#### ALTERNATE WAYS OF VIEWING THE CARRIER-SUBMARINE INTERACTION PROBLEM

##### Discrete Time Intervals

The carrier-submarine interaction problem has been frequently modeled by considering fixed intervals of time  $\Delta t$  so that the "time variable" became the number of time intervals  $(t/\Delta t)$ . The various appropriate probabilities were then raised to this power: such as  $(1 - a\Delta t)^{t/\Delta t}$ . It may be shown that the analytical results of such a model transform directly to the results derived earlier in this paper as the time interval  $\Delta t$  is allowed to become arbitrarily small.

### Unconditional Attrition Rate Constant

It would seem that a model could be developed which assumed the unconditional attrition rate of submarines as constant. This has been done.<sup>1</sup> The results are the same when the carrier is not very likely to be hit and somewhat more optimistic than predicted in the body of this paper when the carrier is likely to be hit. However, this development requires the simultaneous solution of many differential equations and the solutions for any but the simplest cases are complex and difficult to evaluate. A generalization of this development to the case of many carriers operating in the same area has also been completed.<sup>2</sup>

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<sup>1</sup> J. K. Tyson, appendix C.

<sup>2</sup> Reference (h).

## APPENDIX B

## APPENDIX B

### SAMPLE NUMERICAL CALCULATIONS

This appendix illustrates how tactical data can be used to get numerical values for the parameters of the model. Sample calculations are made, and the resulting on-station probabilities and exchange ratios are displayed. To keep the presentation unclassified, the numbers used are arbitrary; they were not derived from exercise data or theoretical studies.<sup>1</sup>

Cases of an escorted carrier opposed by torpedo submarines and by missile submarines, and an unescorted carrier<sup>2</sup> opposed by torpedo submarines are illustrated. An escorted carrier can be detected at longer ranges, owing to the active sonar of its escorts; encounter rates reflect this increased detection range. An unescorted carrier can be attacked immediately once it has been encountered, with virtual assurance that the attack will be successful.

The arbitrary probabilities used to describe the four possible outcomes of submarine-carrier encounters are given in table B-I.

TABLE B-I

Carrier hit		Yes	No
Submarine killed			
Escorted carrier vs. torpedo submarines			
Yes		.21	.29
No		.09	.41
Unescorted carrier vs. torpedo submarines			
Yes		.15	0
No		.85	0
Escorted carrier vs. missile submarines			
Yes		.05	.10
No		.28	.57

<sup>1</sup> Encounter rates were calculated by methods described in reference (a), an unclassified document.

<sup>2</sup> Operating at slow speed and under strict electronic silence in a dispersed formation.

Escorted carrier versus torpedo submarines:

$$\begin{aligned} b &= \text{encounter rate per day} \\ &= .25 \text{ (corresponds to a detection range of 35 miles in} \\ &\quad \text{an operating area of 70,000 square miles)} \\ a &= b \cdot Pr \text{ (submarine killed given that the carrier is not} \\ &\quad \text{hit) + rate of kill from all other causes} \\ &= .25 \cdot .29 + .05 = .12 \\ h &= Pr \text{ (carrier hit, given an encounter)} \\ &= .21 + .09 = .3 \\ S &= \text{unconditional rate of submarine attrition} \\ &= .25 (.21 + .29) + .05 = .18 . \end{aligned}$$

Unescorted carrier versus torpedo submarines:

$$\begin{aligned} b &= .02 \text{ (corresponds to a detection range of 7 miles in an} \\ &\quad \text{area of 70,000 square miles)}^3 \\ a &= .02 (0) + .05 = .05 \\ h &= 1.0 \\ S &= .05 . \end{aligned}$$

Unescorted carrier versus cruise missile submarines: If the carrier is unescorted the submarine must see it to find it and should thus torpedo it. This is therefore identical to the previous case.

Escorted carrier versus cruise missile submarines:

$$\begin{aligned} b &= .25 \\ a &= .25 \cdot .10 + .05 = .075 \\ h &= .33 \\ S &= .09 . \end{aligned}$$

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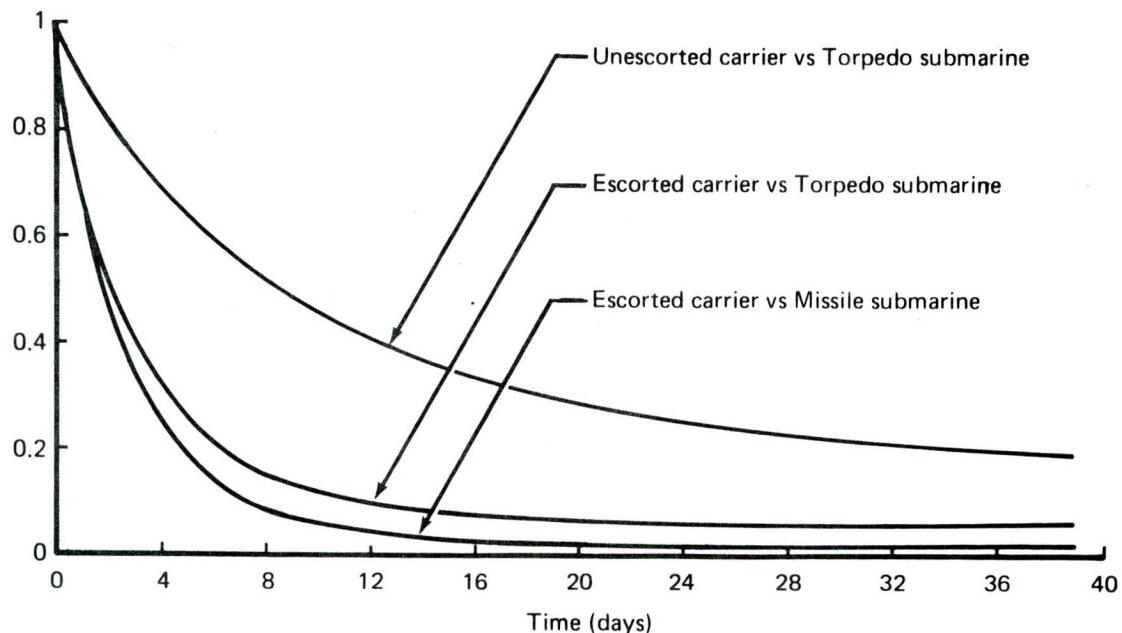
<sup>3</sup> Carrier average SOA 10 knots, submarine average SOA 3 knots. The submarine can identify the carrier only during daylight.

The numerical results of these three examples are given in table B-II, for  $x_0 = 5$  and  $r = 0$  in each case. The probability of zero successful attacks up to time  $t$ ,  $P_0(t)$ , is shown in figure B-1. The probability of the carrier's being on station at time  $t$ ,  $\pi_1(t)$ , is shown in figure B-2, for  $N = 1$ .

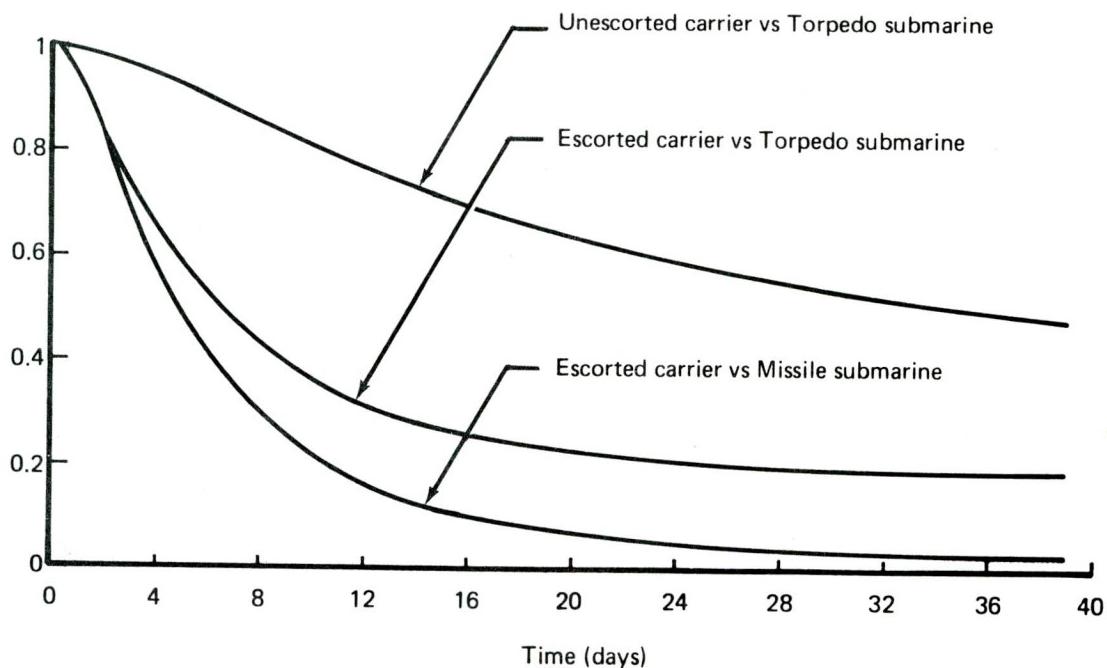
TABLE B-II  
NUMERICAL RESULTS OF SAMPLE CALCULATIONS

	Escorted carrier versus torpedo submarines	Unescorted carrier* versus torpedo submarines	Escorted carrier versus missile submarines
Encounter rate, $b$	.25	.02	.25
Conditional submarine attrition rate, $a$	.12	.05	.075
Overall submarine attrition rate, $S$	.18	.05	.09
Pr (successful attack given encounter), $h$	.3	1.0	.33
Exchange ratio (if $N=0$ ), $R$	2.3	2.7	1.1
Exchange ratio (if $N=1$ ), $R$	4.7	5.3	2.1
Pr (carrier on station after 10 days, $N=0$ ), $\pi_0(10)$	.12	.45	.06
Pr (carrier on station after 10 days, $N=1$ ), $\pi_1(10)$	.36	.81	.21

\* "Quiet" carrier in dispersed formation.



**FIG. B-1: ON-STATION PROBABILITY OF A CARRIER THAT MUST RETIRE THE FIRST TIME IT IS SUCCESSFULLY ATTACKED**



**FIG. B-2: ON-STATION PROBABILITY OF A CARRIER THAT CAN ABSORB ONE SUCCESSFUL ATTACK AND STILL REMAIN OPERATIONAL**

## **APPENDIX C**

## APPENDIX C

### A MODEL OF CARRIER-SUBMARINE INTERACTIONS WITH UNCONDITIONAL ATTRITION RATE CONSTANT

#### INTRODUCTION

The body of this paper describes a comparatively simple model for carrier-submarine interactions which provides, as a function of time, a probability distribution for the number of successful attacks on the carrier and the expected number of submarines surviving the engagement. This appendix addresses the same basic model, but follows a different procedure which avoids certain difficulties.

In brief, the assumptions of this development are as follows:

- a. Each submarine in the area will independently and randomly encounter the carrier and/or its local defenses at a constant rate denoted by "b".
- b. Each encounter will result in one of four possible events, with associated probabilities:
  1. With probability  $P_1$ , the carrier is hit and the submarine is eliminated.
  2. With probability  $P_2$ , the carrier is not hit and the submarine is eliminated.
  3. With probability  $P_3$ , the carrier is hit and the submarine survives the encounter.
  4. With probability  $P_4$ , the carrier is not hit and the submarine survives the encounter.
- c. Each submarine will independently and randomly encounter and be eliminated by area defenses at a constant rate denoted by " $\alpha$ ".
- d. Additional submarines may enter into the engagement at random with a constant rate denoted by "r".

These assumptions define a continuous time Markov chain in which the state of the system is described by the number of "hits" received by the carrier and the number of surviving submarines in the engagement. The state probabilities are denoted by  $f_{n,h}(t)$ , defined as the probability at time  $t$  that the carrier has received  $h$  hits and is engaged by  $n$  surviving submarines. The possible transitions between states, and their associated rates, are shown in the following table:

	State change	Transition rate
n, h	$\rightarrow$	$n-1, h+1$
n, h	$\rightarrow$	$n-1, h$
n, h	$\rightarrow$	$n, h + 1$
n, h	$\rightarrow$	$n + 1, h$

The state probabilities are determined by solving the following set of differential equations:

$$\begin{aligned} \frac{df_{n,h}}{dt} = & -n \left[ b(P_1 + P_2 + P_3) + \alpha \right] f_{n,h} - r f_{n,h} \\ & + (n+1) b P_1 f_{n+1,h-1} + (n+1) \left[ b P_2 + \alpha \right] f_{n+1,h} \\ & + n b P_3 f_{n,h-1} + r f_{n-1,h} . \end{aligned} \quad (C-1)$$

This equation applies for all  $n, h$ , with the understanding that terms involving  $f$  with negative subscripts are to be suppressed. The initial conditions, when  $N$  submarines are engaged at the start, are

$$\begin{aligned} f_{N,o} &= 1 \\ f_{n,h} &= 0 \quad \text{all other } n, h . \end{aligned}$$

#### SPECIAL CASE, ONE OPPOSING SUBMARINE

The set of equations C-1 is simplified somewhat if only one submarine is initially present and there is no reinforcement. Then only the state probabilities  $f_{1,h}$  and  $f_{0,h}$  need be considered, and the set C-1 specializes to

$$\frac{df_{1,o}}{dt} = - [b(P_1 + P_2 + P_3) + \alpha] f_{1,o} \quad (C-2.1)$$

$$\frac{df_{1,h}}{dt} = - [b(P_1 + P_2 + P_3) + \alpha] f_{1,h} + b P_3 f_{1,h-1}$$

$$\frac{df_{o,o}}{dt} = (bP_2 + \alpha) f_{1,o} \quad (C-2.2)$$

$$\frac{df_{o,h}}{dt} = bP_1 f_{1,h-1} + (bP_2 + \alpha) f_{1,h} .$$

The initial conditions are

$$f_{1,o} = 1$$

$$f_{1,h} = f_{o,o} = f_{o,h} = 0 . \quad (C-3)$$

The probability that the carrier receives no hits is, by definition, the sum of  $f_{1,o}$  and  $f_{o,o}$ . To determine  $f_{1,o}$  and  $f_{o,o}$  we integrate equations C-2.1 and C-2.2:

$$f_{1,o} = \exp - [b(P_1 + P_2 + P_3) + \alpha] t$$

$$f_{o,o} = \frac{b P_2 + \alpha}{b(P_1 + P_2 + P_3) + \alpha} \left( 1 - \exp - [b(P_1 + P_2 + P_3) + \alpha] t \right).$$

The probability the carrier receives no hits is now given by

$$P_o = 1 - \frac{b(P_1 + P_3)}{b(P_1 + P_2 + P_3) + \alpha} \left( 1 - \exp - [b(P_1 + P_2 + P_3) + \alpha] t \right) .$$

If  $N$  submarines are initially present and are not reinforced, we would expect the probability the carrier receives no hits to be given by

$$P_o = \left\{ 1 - \frac{b(P_1 + P_3)}{b(P_1 + P_2 + P_3) + \alpha} \left( 1 - \exp - [b(P_1 + P_2 + P_3) + \alpha] t \right) \right\}^N$$

in view of the independence assumption of the model. This result differs from equation 2 of the main body in functional form presumably because the latter is based at the outset on an expected-value calculation of the number of submarines versus time. Of possibly greater significance, the parameter combination

$b(P_1 + P_2 + P_3) + \alpha$  here replaces the expression  $a \equiv bP_2 + \alpha$  used in the main body.

To find the distribution of the number of hits received by the carrier requires the recursive integration of the full set of equations C-2. This is painlessly accomplished by introducing the probability generating function,<sup>1</sup> defined as follows:

$$F_1(X, t) = \sum_{h=0}^{\infty} f_{1,h}(t) X^h$$

$$F_0(X, t) = \sum_{h=0}^{\infty} f_{0,h}(t) X^h .$$

From equations C-2 and C-3 it follows that

$$\frac{dF_1}{dt} = - [b(P_1 + P_2 + P_3) + \alpha] F_1 + X bP_3 F_1$$

$$\frac{dF_0}{dt} = (bP_2 + \alpha) F_1 + X bP_1 F_1$$

$$F_1(X, 0) = 1$$

$$F_0(X, 0) = 0 .$$

The solution of these equations is easily seen to be

$$F_1(X, t) = \exp \{ - [b(P_1 + P_2 + P_3) + \alpha] t + X bP_3 t \}$$

$$F_0(X, t) = \frac{bP_2 + \alpha + XbP_1}{b(P_1 + P_2 + P_3) + \alpha - XbP_3}$$

$$[ 1 - \exp \{ - [b(P_1 + P_2 + P_3) + \alpha] + XbP_3 t \} ] .$$

The generating function for the probability that the carrier receives exactly  $h$  hits is just the sum of  $F_1$  and  $F_0$ . If  $N$  submarines are present and are not reinforced, we would expect the generating function to be  $(F_1 + F_0)^N$ , in view of the independence assumption of the model. Further, the generating function

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<sup>1</sup> W. Feller, "An Introduction to Probability Theory and its Application," Vol. 1, p. 264.

for the probability that the carrier receives  $h$  or fewer hits is given by<sup>2</sup>

$$\sum_{h=0}^{\infty} \pi_h(t) X^h = \frac{[F_1(X, t) + F_0(X, t)]^N}{1-X},$$

where the left-hand side employs the notation given in the main body. This result reduces the problem of finding the  $\pi_h$ 's to a routine but tedious process of power series expansions. For example, when  $N = 1$ , and in the limit  $t \rightarrow \infty$ ,

$$P_0 = \pi_0 = \frac{bP_2 + \alpha}{b(P_1 + P_2 + P_3) + \alpha},$$

and if  $h \geq 1$ ,

$$P_h = \frac{[b(P_1 + P_2) + \alpha] [b(P_1 + P_3)] (bP_3)^{h-1}}{[b(P_1 + P_2 + P_3) + \alpha]^{h+1}}$$

$$\pi_h = 1 - \left( \frac{bP_3}{b(P_1 + P_2 + P_3) + \alpha} \right)^h.$$

Finally, the probability that the submarine survives for time  $t$  is given by  $F_1(1, t) = \exp - [b(P_1 + P_2) + \alpha] t$ .

#### GENERAL CASE

The solution of the more general equations C-1 can be carried out by a process similar to that described above. A generating function for the joint distribution of the two random variables  $n$  and  $h$  is defined as follows:

$$F(Z, X, t) = \sum_{n, h=0}^{\infty} f_{n, h} Z^n X^h.$$

The equation governing  $F$  is determined by multiplying each side of equation

<sup>2</sup> Op. cit., p. 265.

C-1 by  $Z^n X^h$  and summing. The final result, on making use of identities such as

$$nZ^n \equiv \frac{Z \partial Z^n}{\partial Z} ,$$

is found to be

$$\begin{aligned} & \frac{\partial F}{\partial t} + r(1-Z)F \\ & + \{ [(P_1 + P_2 + P_3(1-X)) + \alpha] Z - [b(P_1 X + P_2) + \alpha] \} \frac{\partial F}{\partial Z} = 0 , \end{aligned} \quad (C-4)$$

with initial condition, describing the presence of  $N$  submarines and no hits received by the carrier, given by

$$F(Z, X, 0) = Z^N . \quad (C-5)$$

For convenience, the following "dimensionless" parameters are introduced:

$$\begin{aligned} \tau &= [b(P_1 + P_2 + P_3) + \alpha - XbP_3] t \\ \rho &= \frac{r}{[b(P_1 + P_2 + P_3) + \alpha - XbP_3]} \\ \sigma &= \frac{bP_2 + \alpha + XbP_1}{b(P_1 + P_2 + P_3) + \alpha - XbP_3} . \end{aligned} \quad (C-6)$$

Equation C-4 then takes the form

$$\frac{\partial F}{\partial \tau} + \rho(1-Z)F + (Z - \sigma) \frac{\partial F}{\partial Z} = 0 . \quad (C-7)$$

A "steady state" solution is defined by

$$\rho(1-Z)Y + (Z - \sigma) \frac{dY}{dZ} = 0$$

$$\begin{aligned} \text{or } Y(Z) &= \exp - \int^Z \rho \frac{(1-Z') dZ'}{Z' - \sigma} \\ &= (Z - \sigma)^{-\rho(1-\sigma)} e^{\rho(Z-\sigma)} . \end{aligned}$$

Equation C-7 is transformed to a more convenient form by making a substitution,  $F(Z, t) = \gamma(Z) G(Z, t)$ , which leads to the following equation for  $G$ :

$$\frac{\partial G}{\partial \tau} + (Z - \sigma) \frac{\partial G}{\partial Z} = 0. \quad (C-8)$$

The general solution of equation C-8 is an arbitrary function of

$$\tau - \int^Z \frac{dZ'}{Z' - \sigma}$$

or, equivalently, a function of  $(Z - \sigma)e^{-\tau}$ . Thus, the general solution of equation C-7 is given by

$$F(Z, \tau) = (Z - \sigma)^{-\rho(1-\sigma)} e^{-\rho(Z-\sigma)} G[(Z-\sigma)e^{-\tau}].$$

The functional form of  $G$  is established by making use of the initial condition, equation C-5. The result of this process is the solution

$$F(Z, X, t) = e^{-\rho[(Z-\sigma)(1-e^{-\tau}) - (1-\sigma)\tau]} \left[ \sigma + (Z-\sigma)e^{-\tau} \right]^N, \quad (C-9)$$

where the parameters  $\rho, \sigma, \tau$  are functions of  $X$  as defined by equations C-6.

Particular results can be derived from C-9 as desired. For example, the expected number of submarines at time  $t$  is given by

$$\begin{aligned} \bar{n} &= \frac{\partial F}{\partial Z} \Big|_{Z=1}^{X=1} \\ &= \frac{r}{b(P_1 + P_2) + \alpha} \left[ 1 - e^{-[b(P_1 + P_2) + \alpha]t} \right] + Ne^{-[b(P_1 + P_2) + \alpha]t}. \end{aligned}$$

The distribution of the number of hits received by the carrier has the generating function given by

$$\begin{aligned} F(1, X, t) &= \left\{ \frac{bP_2 + \alpha + XbP_1 + b(P_1 + P_3)(1-X)e^{-[b(P_1 + P_2 + P_3) + \alpha]t + XbP_3 t}}{b(P_1 + P_2 + P_3) + \alpha - XbP_3} \right\}^N \\ &\times \exp \frac{rb(P_1 + P_3)(1-X)}{(b(P_1 + P_2 + P_3) + \alpha - XbP_3)^2} \\ &\left[ 1 - [b(P_1 + P_2 + P_3) + \alpha]t + XbP_3 t - e^{-[b(P_1 + P_2 + P_3) + \alpha]t + XbP_3 t} \right]. \end{aligned} \quad (C-10)$$

The probability that the carrier receives exactly  $h$  hits is obtained by finding the coefficient of  $X^h$  in the right-hand side of equation C-10. The calculations are straightforward but lengthy. For convenience in presentation of results, the probability distribution will be given as the convolution of two distributions:

1. The probability of  $h$  hits, given that  $N$  submarines are engaged initially and there is no reinforcement:  $P_h(t|N, 0)$ .
2. The probability of  $h$  hits, given that no submarines are engaged initially and that submarines are introduced at a rate  $r$ :  $P_h(t|0, r)$ .

The following definitions are employed:

$$\lambda = \frac{b(P_1 + P_3)}{b(P_1 + P_2 + P_3) + \alpha}$$

$$\rho = \frac{rb(P_1 + P_3)}{[b(P_1 + P_2 + P_3) + \alpha]^2}$$

$$\zeta = \frac{bP_3}{b(P_1 + P_2 + P_3) + \alpha}$$

$$\tau = [b(P_1 + P_2 + P_3) + \alpha] t$$

$$Q_J(\tau) = 1 - \sum_{k=0}^J \frac{\tau^k}{k!} e^{-\tau}.$$

The quantities  $Q_J$  are the "tails" of a Poisson distribution with a parameter  $\tau$ ; equivalently, they are the probability that the sum of  $(J+1)$  exponentially

distributed variables has a value less than  $\tau$ . With these definitions, the results for the first few  $P_h$ 's have the following explicit forms:

$$P_0(t|N, 0) = (1-\lambda Q_0)^N$$

$$P_1(t|N, 0) = N \lambda (1-\lambda Q_0)^{N-1} (Q_0 - \zeta Q_1)$$

$$P_2(t|N, 0) = \binom{N}{2} \lambda^2 (1-\lambda Q_0)^{N-2} (Q_0 - \zeta Q_1)^2 + N \lambda \zeta (1-\lambda Q_0)^{N-1} (Q_1 - \zeta Q_2)$$

$$P_3(t|N, 0) = \binom{N}{3} \lambda^3 (1-\lambda Q_0)^{N-3} (Q_0 - \zeta Q_1)^3$$

$$+ 2 \binom{N}{2} \lambda^2 \zeta (1-\lambda Q_0)^{N-2} (Q_0 - \zeta Q_1) (Q_1 - \zeta Q_2)$$

$$+ N \lambda \zeta^2 (1-\lambda Q_0)^{N-1} (Q_2 - \zeta Q_3)$$

$$P_4(t|N, 0) = \binom{N}{4} \lambda^4 (1-\lambda Q_0)^{N-4} + 3 \binom{N}{3} \lambda^2 \zeta (1-\lambda Q_0)^{N-3} (Q_0 - \zeta Q_1)^2 (Q_1 - \zeta Q_2)$$

$$+ \binom{N}{2} \lambda^2 \zeta^2 (1-\lambda Q_0)^{N-2} \left\{ (Q_1 - \zeta Q_2)^2 + 2(Q_0 - Q_1)(Q_2 - Q_3) \right\}$$

$$+ N \lambda \zeta^3 (1-\lambda Q_0)^{N-1} (Q_3 - \zeta Q_4)$$

$$P_0(t|0, r) = e^{-\rho(\tau-Q_0)}$$

$$P_1(t|0, r) = e^{-\rho(\tau-Q_0)} \rho [(1-\zeta) \tau Q_0 - (1-2\zeta) Q_1]$$

$$P_2(t|0, r) = e^{-\rho(\tau-Q_0)} \left\{ \frac{\rho^2}{2!} \left[ (1-\zeta) \tau Q_0 - (1-2\zeta) Q_1 \right]^2 + \rho \zeta \left[ (1-\zeta) \tau Q_1 - (2-3\zeta) Q_2 \right] \right\}$$

$$P_3(t|0, r) = e^{-\rho(\tau-Q_0)} \left\{ \frac{\rho^3}{3!} \left[ (1-\zeta) \tau Q_0 - (1-2\zeta) Q_1 \right]^3 \right.$$

$$+ \rho^2 \zeta \left[ (1-\zeta) \tau Q_0 - (1-2\zeta) Q_1 \right] \left[ (1-\zeta) \tau Q_1 - (2-3\zeta) Q_2 \right]$$

$$\left. + \rho \zeta^2 \left[ (1-\zeta) \tau Q_2 - (3-4\zeta) Q_3 \right] \right\}$$

$$\begin{aligned}
P_4(t|0, r) = & e^{-\rho} (\tau - Q_0) \left\{ \frac{\rho^4}{4!} \left[ (1-\zeta) \tau Q_0 - (1-2\zeta) Q_1 \right]^4 \right. \\
& + 1/2 \rho^3 \zeta \left[ (1-\zeta) \tau Q_0 - (1-2\zeta) Q_1 \right]^2 \left[ (1-\zeta) \tau Q_1 - (2-3\zeta) Q_2 \right] \\
& + \rho^2 \zeta^2 \left[ (1-\zeta) \tau Q_0 - (1-2\zeta) Q_1 \right] \left[ (1-\zeta) \tau Q_2 - (3-4\zeta) Q_3 \right] \\
& \left. + 1/2 \rho^2 \zeta^2 \left[ (1-\zeta) \tau Q_1 - (2-3\zeta) Q_2 \right]^2 + \rho \zeta^3 \left[ (1-\zeta) \tau Q_3 - (4-5\zeta) Q_4 \right] \right\}.
\end{aligned}$$

In view of the complexity of these expressions, it seems worth pointing out one special case. If it is justified to assume that the carrier can survive  $h$  hits with probability  $q^h$ , then a comparatively simple expression for the probability that the carrier survives for time  $t$  is provided by equation C-10 with the input  $X=q$ . In the notation used directly above the result is

$$\text{Probability of survival} = \left\{ e^{\frac{-\rho(1-q)}{(1-\zeta q)^2} \left[ \tau(1-\zeta q) - 1 + e^{-\tau(1-\zeta q)} \right]} \right\} \times \left[ 1 - \frac{\lambda(1-q)}{1-\zeta q} \left( 1 - e^{-\tau(1-\zeta q)} \right) \right]^N.$$

This result, of course, can be derived directly from the basic model with appropriate modification of the probabilities describing the result of an engagement.

#### POISSON APPROXIMATION

Returning to equation C-10 and the dimensionless parameters defined after, we note that if the parameter  $\zeta$  is set equal to zero the probability distribution reduces exactly to a convolution of Poisson and binomial random variables. This parameter appears to account for multiple successful attacks by an individual submarine. This observation, together with the fact that a binomial distribution can be approximated by a Poisson distribution, suggests that the total hit distribution can be approximated by a Poisson distribution – indeed, this is the result derived in this paper. There are various ways to obtain such a representation; the most straightforward is to use the generating function, equation C-10.

The generating function is written in the form  $F(1, X, t) = \exp \Phi(X)$ , where, of course,  $\Phi(X)$  is the logarithm of the right-hand side of equation C-10. The coefficients  $\Phi_R$  are defined by the power series expansion of  $\Phi(X)$ . Thus,

$$F(1, X, t) = \exp [\Phi_0 + \Phi_1 X + \Phi_2 X^2 + \dots] .$$

The next step is to expand  $\left[ \exp \Phi_2 X^2 + \Phi_3 X^3 + \dots \right]$  as a power series; after some rearrangement the result becomes

$$F(1, X, t) = e^{\Phi_0 + \Phi_1} e^{\Phi_1 (X-1)} \left[ 1 + \Phi_2 X^2 + \Phi_3 X^3 + \left( \frac{\Phi_2^2}{2} + \Phi_4 \right) X^4 + \dots \right]. \quad (C-11)$$

Equation C-11 demonstrates that the distribution of hits can be expressed as a sequence of Poisson probabilities multiplied by a normalization factor. For example, the cumulative distribution of hits,  $\pi_h$ , can be written as

$$\pi_h = A \left[ S_h(\Phi) + \Phi_2 S_{h-2}(\Phi) + \Phi_3 S_{h-3}(\Phi_1) + \dots \right], \quad (C-12)$$

where  $A \equiv \exp(\Phi_0 + \Phi_1)$  and  $S_h$  are the cumulative probabilities of a Poisson distribution having a parameter  $\Phi_1$ . This result is exact if the expansion is carried out to the final term in  $S_0$ ; whether or not the first few terms are a good approximation is determined by the magnitude of the higher-order coefficients  $\Phi_2$ ,  $\Phi_3$ , etc. The parameters in equation C-12 take the following form:

$$\begin{aligned} A &= (1 - \lambda Q_0)^N \exp \left\{ \frac{N\lambda}{1 - \lambda Q_0} (Q_0 - \zeta Q_1) - \zeta (\tau Q_0 - 2Q_1) \right\} \\ \Phi_1 &= \frac{N\lambda}{1 - \lambda Q_0} (Q_0 - \zeta Q_1) + \rho \left[ (\tau - Q_0) - \zeta (\tau Q_0 - 2Q_1) \right] \\ \Phi_2 &= \frac{N\lambda}{1 - \lambda Q_0} \zeta (Q_1 - \zeta Q_2) - \frac{1/2 N\lambda^2}{(1 - \lambda Q_0)^2} (Q_0 - \zeta Q_1)^2 + \rho \zeta \left[ (\tau Q_0 - 2Q_1) - \zeta (\tau Q_1 - 3Q_2) \right] \\ \Phi_3 &= \frac{N\lambda}{1 - \lambda Q_0} \zeta^2 (Q_2 - \zeta Q_3) - \frac{N\lambda^2}{(1 - \lambda Q_0)^2} \zeta (Q_0 - \zeta Q_1) (Q_1 - \zeta Q_2) + \frac{1/3 N\lambda^3}{(1 - \lambda Q_0)^3} (Q_0 - \zeta Q_1)^3 \\ &\quad + \rho \zeta^2 \left[ (\tau Q_1 - 3Q_2) - \zeta (\tau Q_2 - 4Q_3) \right] . \end{aligned}$$

From these expressions it can be seen that if  $N\lambda^2 \ll 1$  and  $\zeta \ll 1$ , then  $A \rightarrow 1$  and  $\Phi_2, \Phi_3$ , etc.,  $\rightarrow 0$ , and equation C-12 reduces to the form given in the main body.

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13. ABSTRACT This paper develops analytic expressions for the probability that a single aircraft carrier, opposed by submarines, can remain on station for a specified length of time. The threat from both torpedo and cruise-missile submarines is considered, but the threat from aircraft is not. Expressions for expected submarine losses are derived. Expressions are also developed to show how the probability of remaining on station improves as the carrier's resistance to damage increases. A wide variety of ASW force compositions and tactics can be represented in the parameters of the model. Appendix C presents an alternative way to formulate the problem and derive the quantities mentioned above. The two methods produce results that are in excellent agreement. The intention of the authors is to provide a tool that can be used to improve carrier effectiveness through the study of tactics and force interactions.		
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